CHAIN MIXING ENDOMORPHISMS ARE APPROXIMATED BY SUBSHIFTS ON THE CANTOR SET

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Abstract. Let f be a chain mixing continuous onto mapping from the Cantor set onto itself. Let g be a homeomorphism on the Cantor set that is topologically conjugate to a subshift. Then, homeomorphisms that are topologically conjugate to g approximate f in the topology of uniform convergence if a trivial necessary condition on the periodic points holds. In particular, if f is a chain mixing continuous onto mapping from the Cantor set onto itself with a fixed point, then homeomorphisms on the Cantor set that are topologically conjugate to a subshift approximate f in the topology of uniform convergence. In addition, homeomorphisms on the Cantor set that are topologically conjugate to a subshift without periodic points approximate any chain mixing continuous onto mappings from the Cantor set onto itself. In particular, let f be a homeomorphism on the Cantor set that is topologically conjugate to a full shift. Let g be a homeomorphism on the Cantor set that is topologically conjugate to a subshift. Then, a sequence of homeomorphisms that is topologically conjugate to g approximates f.

1. Introduction

Let (X, d) be a compact metric space. Let $f : X \to X$ be a continuous onto mapping. In this manuscript, the pair (X, f) is called a *topological dynamical system*. Let $\mathcal{H}^+(X)$ be the set of all topological dynamical systems on X. For any f and g in $\mathcal{H}^+(X)$, we define $d(f, g) := \sup_{x \in X} d(f(x), g(x))$. Then, $(\mathcal{H}^+(X), d)$ is a metric space of uniform convergence. $\mathcal{H}(X)$ denotes the set of all homeomorphisms from X onto itself. In this manuscript, we mainly consider the case in which X is homeomorphic to the Cantor set, denoted by C. T. Kimura [3, Theorem 1] and I [4] have shown that the subset of $\mathcal{H}(C)$ consisting of all expansive homeomorphisms with the pseudo-orbit tracing property is dense in $\mathcal{H}(C)$. $\mathrm{SFT}(C)$ denotes the set of all $f \in \mathcal{H}(C)$ that is topologically conjugate to some two-sided subshift of finite type. Then, $\mathrm{SFT}(C)$ coincides with the set of all expansive $f \in \mathcal{H}(C)$ with the pseudo-orbit tracing property (P. Walters [5, Theorem 1]). Therefore, $\mathrm{SFT}(C)$ is dense in $\mathcal{H}(C)$. A topological dynamical system (X, f) is said to be *topologically mixing* if for any pair of non-empty open sets $U, V \subset X$, there exists a non-negative integer N such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$. In [4], it is shown that if $f \in \mathcal{H}$ is topologically mixing, then there exists a sequence $\{g_k\}_{k=1,2,\dots}$ of topologically mixing elements of $\mathrm{SFT}(C)$ such that

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$g_k \to f$ as $k \to \infty$. Let f be a chain mixing element of $\mathcal{H}^+(C)$ and g, an element of $\mathcal{H}(C)$ that is topologically conjugate to a two-sided subshift. In this manuscript, we consider the condition in which homeomorphisms that are topologically conjugate to g approximate f. Let (X, f) be a topological dynamical system and $\delta > 0$. A sequence $\{x_i\}_{i=0,1,\ldots,l}$ of elements of X is a *δ chain* from x_0 to x_l if $d(f(x_i), x_{i+1}) < \delta$ for all $i = 0, 1, \ldots, l-1$. Then, l is called the length of the chain. A topological dynamical system (X, f) is *chain mixing* if for every $\delta > 0$ and every pair $x, y \in X$, there exists a positive integer N such that for all $n > N$, there exists a δ chain from x to y of length n. Let (X, f) and (Y, g) be topological dynamical systems. We write $(Y, g) \rhd (X, f)$ if there exists a sequence of homeomorphisms $\{\psi_k\}_{k=1,2,\ldots}$ from Y onto X such that $\psi_k \circ g \circ {\psi_k}^{-1} \to f$ as $k \to \infty$. If $(Y, g) \rhd (X, f)$ and if g^n has a fixed point for some positive integer n, then f^n must also have a fixed point. We write $(Y, g) \xrightarrow{\text{per}} (X, f)$ if this trivial necessary condition on periodic points holds. We show the following:

Theorem 1.1. *Let X be homeomorphic to the Cantor set. Let (X, f) be a chain mixing topological dynamical system. Let (Λ, σ) be a two-sided subshift such that Λ is homeomorphic to C. Then, the following conditions are equivalent:*

(1) $(\Lambda, \sigma) \xrightarrow{\text{per}} (X, f)$;
(2) $(\Lambda, \sigma) \rhd (X, f)$.

Corollary 1.2. *Let X be homeomorphic to the Cantor set. Let (X, f) be a chain mixing topological dynamical system with a fixed point. Let (Λ, σ) be a two-sided subshift such that Λ is homeomorphic to C. Then, $(\Lambda, \sigma) \rhd (X, f)$.*

Corollary 1.3. *Let X be homeomorphic to the Cantor set. Let (X, f) be a chain mixing topological dynamical system. Let (Λ, σ) be a two-sided subshift such that Λ is homeomorphic to C without periodic points. Then, $(\Lambda, \sigma) \rhd (X, f)$.*

Corollary 1.4. *Let $n > 1$ be an integer. Let (Σ_n, σ) be the two-sided full shift of n symbols. Let (Λ, σ) be a two-sided subshift such that Λ is homeomorphic to C. Then, $(\Lambda, \sigma) \rhd (\Sigma_n, \sigma)$.*

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2. Preliminaries

Let $\mathbf{Z}$ denote the set of all integers; $\mathbf{N}$, the set of all nonnegative integers; and $\mathbf{Z}_+$, the set of all positive integers. Let (X, d) be a compact metric space. For any two maps f and g from X to itself, we define $d(f, g) := \sup\{d(f(x), g(x)) \mid x \in X\}$.

Let $f : X \to X$ be a continuous onto mapping. In this manuscript, (X, f) is called a *topological dynamical system*. A topological dynamical system (X, f) is *topologically conjugate* to a topological dynamical system (Y, g) if there exists a homeomorphism $\psi : Y \to X$ such that $f \circ \psi = \psi \circ g$. Such

a homeomorphism is called a *topological conjugacy*. In this manuscript, we write $(Y, g) \rhd (X, f)$ if there exists a sequence of homeomorphisms $\{\psi_k\}_{k=1,2,\dots}$ from Y onto X such that $\psi_k \circ g \circ {\psi_k}^{-1} \to f$ as $k \to \infty$, i.e. $d(\psi_k \circ g \circ {\psi_k}^{-1}, f) \to 0$ as $k \to \infty$.

Lemma 2.1. *Let (X, f) be a topological dynamical system. Let (Y_k, g_k) $(k = 1, 2, \dots)$ be a sequence of topological dynamical systems. Suppose that there exists a sequence of homeomorphisms $\psi_k : Y_k \to X$ such that $\psi_k \circ g_k \circ {\psi_k}^{-1} \to f$ as $k \to \infty$. Let (Z, h) be a topological dynamical system such that $(Z, h) \rhd (Y_k, g_k)$ for all $k = 1, 2, \dots$. Then, $(Z, h) \rhd (X, f)$.*

Proof. Let $\epsilon > 0$. Then, there exists $N \in \mathbf{Z}_+$ such that $d(\psi_k \circ g_k \circ {\psi_k}^{-1}, f) < \epsilon/2$ for all $k > N$. Assume $k > N$. Let $\delta > 0$ be such that if $d(y, y') < \delta$, then $d(\psi_k(y), \psi_k(y')) < \epsilon/2$. Because $(Z, h) \rhd (Y_k, g_k)$, there exists a homeomorphism $\psi' : Z \to Y_k$ such that $d(\psi' \circ h \circ {\psi'}^{-1}, g_k) < \delta$. Therefore, we find that $d((\psi_k \circ \psi') \circ h \circ (\psi_k \circ \psi')^{-1}, f) < d(\psi_k \circ (\psi' \circ h \circ \psi'^{-1}) \circ \psi_k^{-1}, \psi_k \circ g_k \circ {\psi_k}^{-1}) + d(\psi_k \circ g_k \circ {\psi_k}^{-1}, f) < \epsilon$. □

For a topological dynamical system (X, f), we define

$$\mathrm{Per}(X, f) := \{n \in \mathbf{Z}_+ \mid f^n(x) = x \text{ for some } x \in X\}.$$

Let (X, f) and (Y, g) be topological dynamical systems. Suppose that $(Y, g) \rhd (X, f)$. Then, for each $n \in \mathbf{Z}_+$, $(Y, g^n) \rhd (X, f^n)$. Consider a sequence of homeomorphisms $\{\psi_k\}_{k=1,2,\dots}$ from Y onto X such that $\psi_k \circ g \circ {\psi_k}^{-1} \to f$ as $k \to \infty$. Then, for each $n \in \mathbf{Z}_+$, the fixed points of $\psi_k \circ g^n \circ {\psi_k}^{-1}$ approach some of the fixed points of f^n. Thus, we obtain $\mathrm{Per}(Y, g) \subset \mathrm{Per}(X, f)$. We write $(Y, g) \xrightarrow{\text{per}} (X, f)$ if $\mathrm{Per}(Y, g) \subset \mathrm{Per}(X, f)$. Thus, we obtain the following:

Lemma 2.2. *Let (X, f) and (Y, g) be topological dynamical systems. If $(Y, g) \rhd (X, f)$, then $(Y, g) \xrightarrow{\text{per}} (X, f)$.*

Let C be the Cantor set in the interval $[0, 1]$. A compact metrizable totally disconnected perfect space is homeomorphic to C. Therefore, any non-empty open and closed subset of C is homeomorphic to C. Let $V = \{v_1, v_2, \dots, v_n\}$ be a finite set of n symbols with discrete topology. Let $\Sigma(V) := V^{\mathbf{Z}}$ with the product topology. Then, $\Sigma(V)$ is a compact metrizable totally disconnected perfect space, and hence, it is homeomorphic to C. We define a homeomorphism $\sigma : \Sigma(V) \to \Sigma(V)$ as follows:

$$\sigma(x)_i = x_{i+1} \text{ for all } i \in \mathbf{Z}.$$

The pair $(\Sigma(V), \sigma)$ is called a *two-sided full shift* of n symbols. If a closed set $\Lambda \subset \Sigma(V)$ is invariant under σ, i.e. $\sigma(\Lambda) = \Lambda$, then $(\Lambda, \sigma|_\Lambda)$ is called a *two-sided subshift*. In this manuscript, $\sigma|_\Lambda$ is abbreviated to σ. A directed graph G is a pair (V, E) of a finite set V of vertices and a set of directed edges $E \subset V \times V$. Let $G = (V, E)$ be a directed graph. $\Sigma(G)$ denotes the two-sided subshift defined as follows:

$$\Sigma(G) := \{x \in V^{\mathbf{Z}} \mid (x_i, x_{i+1}) \in E \text{ for all } i \in \mathbf{Z}\}.$$

A two-sided subshift is said to be of *finite type* if it is topologically conjugate to $(\Sigma(G), \sigma)$ for some directed graph G. Throughout this manuscript, unless

otherwise stated, we assume that all the vertices appear in some element of $\Sigma(G)$, i.e. all the vertices of G have both at least one indegree and at least one outdegree. We define a set of words of length k in $\Sigma(G)$ as follows:

$$W(k, G) := \{w_0 w_1 \cdots w_{k-1} \in V^{\{0,1,\dots,k-1\}} \mid (w_i, w_{i+1}) \in E \text{ for all } i = 0, 1, \dots, k-2\}.$$

For a word $w = a_0 a_1 \dots a_{k-1}$ of length k and an integer m, we define a subset $C_m(w) \subset \Sigma(G)$ as follows:

$$C_m(w) = \{x \in \Sigma(G) \mid x_{m+i} = a_i \text{ for all } i = 0, 1, \dots, k-1\}.$$

Such a set is called a *cylinder*. Because $C_m(w)$ is an open and closed subset of $\Sigma(G)$, if $\Sigma(G)$ is homeomorphic to C and if $C_m(w)$ is not empty, then $C_m(w)$ is also homeomorphic to C. A word $a_0 a_1 \dots a_{k-1} \in W(k, G)$ is also called a *path* of length $k-1$ from a_0 to a_{k-1} in G. Let x be an element of some two-sided subshift. Let $i \le j$ be integers. Then, a word $x_i \dots x_j$ is also called a *segment* of length $j - i + 1$.

Lemma 2.3 (Lemma 1.3 of R. Bowen [1]). *Let $G = (V, E)$ be a directed graph. Suppose that every vertex of V has both at least one outdegree and at least one indegree. Then, $\Sigma(G)$ is topologically mixing if and only if there exists an $N \in \mathbf{Z}_+$ such that for any pair of vertices u and v of V, there exists a path from u to v of length $n > N$.*

Proof. See Lemma 1.3 of R. Bowen [1]. □

Let $f : X \to X$ be a mapping and $\mathscr{U}$, a covering of X. For the sake of conciseness, we define a directed graph $G_{f,\mathscr{U}} = (V_{f,\mathscr{U}}, E_{f,\mathscr{U}})$ as follows:

$$V_{f,\mathscr{U}} = \mathscr{U} \text{ and}$$

$$(a_0, a_1) \in E_{f,\mathscr{U}} \text{ if } f(a_0) \cap a_1 \neq \emptyset.$$

Note that if $\emptyset \notin \mathscr{U}$, then all the vertices have at least one outdegree. In addition, if f is an onto mapping, then all the vertices have at least one indegree. Let (X, d) be a compact metric space and $K \subset X$. The diameter of K is defined by $\operatorname{diam}(K) := \sup\{d(x, y) \mid x, y \in K\}$. For a finite covering $\mathscr{U}$ of X, we define $\operatorname{mesh}(\mathscr{U}) := \max\{\operatorname{diam}(U) \mid U \in \mathscr{U}\}$.

Lemma 2.4. *Let (X, d) be a compact metric space and $f : X \to X$, a continuous mapping. Then, for any $\epsilon > 0$, there exists $\delta = \delta(f, \epsilon) > 0$ such that*

$$\delta < \frac{\epsilon}{2};$$

$$\textit{if } d(x, y) \le \delta, \textit{ then } d(f(x), f(y)) < \frac{\epsilon}{2} \textit{ for all } x, y \in X.$$

Proof. This lemma directly follows from the uniform continuity of f. □

For two directed graphs $G = (V, E)$ and $G' = (V', E')$, G is said to be a *subgraph* of G' if $V \subseteq V'$ and $E \subseteq E'$.

Lemma 2.5. *Let (X, d) be a compact metric space, $f : X \to X$ be a continuous mapping, and $\epsilon > 0$. Let $\delta = \delta(f, \epsilon)$ be as in lemma 2.4 and $\mathscr{U}$, a finite covering of X such that $\operatorname{mesh}(\mathscr{U}) < \delta$. Let $g : X \to X$ be a mapping such that $G_{g,\mathscr{U}}$ is a subgraph of $G_{f,\mathscr{U}}$. Then, $d(f, g) < \epsilon$.*

Proof. Let $x \in X$. Then, $x \in U$ and $g(x) \in U'$ for some $U, U' \in \mathscr{U}$. Because $G_{g,\mathscr{U}}$ is a subgraph of $G_{f,\mathscr{U}}$, there exists a $y \in U$ such that $f(y) \in U'$. Therefore, from lemma 2.4, it follows that

$$d(f(x), g(x)) \leq d(f(x), f(y)) + d(f(y), g(x)) < \frac{\epsilon}{2} + \mathrm{diam}(U') < \epsilon.$$

□

From this lemma, we obtain the following:

Lemma 2.6. *Let (X, d) be a compact metric space; $f : X \to X$, a continuous mapping; and $\{\mathscr{U}_k\}_{k=1,2,\dots}$, a sequence of coverings of X such that $\mathrm{mesh}(\mathscr{U}_k) \to 0$ as $k \to \infty$. Let $\{g_k\}_{k=1,2,\dots}$ be a sequence of mappings from X to X such that $G_{g_k,\mathscr{U}_k}$ is a subgraph of $G_{f,\mathscr{U}_k}$ for all k. Then, $g_k \to f$ as $k \to \infty$.*

A covering $\mathscr{U}$ of X is called a partition if $U \cap U' = \emptyset$ for all $U, U' \in \mathscr{U}$, where $U \neq U'$. The Cantor set has a partition by open and closed subsets of an arbitrarily small mesh.

Lemma 2.7. *Let $G = (V, E)$ be a directed graph. Suppose that every vertex of G has both at least one outdegree and at least one indegree. Suppose that $\Sigma(G)$ is topologically mixing and that $\Sigma(G)$ is not a single point. Then, $\Sigma(G)$ is homeomorphic to C.*

Proof. Suppose that $\Sigma(G)$ is topologically mixing. Then, by lemma 2.3, there exists an $N \in \mathbf{Z}_+$ such that for any pair u and v of vertices of G, there exists a path from u to v of length n for all $n > N$. Then, it is easy to check that every point $x \in \Sigma(G)$ is not isolated. Hence, $\Sigma(G)$ is homeomorphic to C. □

3. Proof of the main result

In this section, we prove certain lemmas and propositions in order to prove the main result. For a mapping $\pi : Y \to X$ and a covering $\mathscr{U}$ of X, the covering $\{\pi^{-1}(U) \mid U \in \mathscr{U}\}$ is denoted by $\pi^{-1}(\mathscr{U})$. For any mapping $g : Y \to Y$, we define a directed graph $G_{g,\pi,\mathscr{U}} = (V, E)$ as follows:

$$V = \mathscr{U};$$

$$E = \{(a_0, a_1) \in \mathscr{U} \times \mathscr{U} \mid \pi(g(\pi^{-1}(a_0))) \cap a_1 \neq \emptyset\}.$$

A vertex a in $G_{g,\pi,\mathscr{U}}$ has at least one outdegree if $\pi^{-1}(a) \neq \emptyset$.

Lemma 3.1. *Let X and Y be homeomorphic to C. Let $f : X \to X$ be a continuous mapping; $g : Y \to Y$, a mapping; and $\mathscr{U}_k$, a sequence of finite partitions of X by non-empty open and closed subsets such that $\mathrm{mesh}(\mathscr{U}_k) \to 0$ as $k \to \infty$. Suppose that there exists a sequence π_k $(k = 1, 2, \dots)$ of continuous mappings from Y to X such that $\pi_k(Y) \cap U \neq \emptyset$ for all $U \in \mathscr{U}_k$ and that the directed graph $G_{g,\pi_k,\mathscr{U}_k}$ is a subgraph of $G_{f,\mathscr{U}_k}$ for all $k \in \mathbf{Z}_+$. Then, there exists a sequence ψ_k $(k = 1, 2, \dots)$ of homeomorphisms from Y onto X such that $\psi_k \circ g \circ {\psi_k}^{-1} \to f$ as $k \to \infty$.*

Proof. Let $k \in \mathbf{Z}_+$ be fixed. By assumption, for each $U \in \mathscr{U}_k$, $\pi_k^{-1}(U)$ is a non-empty open and closed subset of Y. Therefore, there exists a homeomorphism $\psi_k : Y \to X$ such that $\psi_k(\pi_k^{-1}(U)) = U$ for each $U \in \mathscr{U}_k$. By construction, $G_{\psi_k \circ g \circ \psi_k{}^{-1}, \mathscr{U}_k} = G_{g, \pi_k, \mathscr{U}_k}$. By assumption, $G_{\psi_k \circ g \circ \psi_k{}^{-1}, \mathscr{U}_k}$ is a subgraph of $G_{f, \mathscr{U}_k}$. Therefore, the conclusion follows from lemma 2.6. □

Lemma 3.2. *Let X and Y be homeomorphic to C. Let $f : X \to X$ be a continuous mapping; $g : Y \to Y$, a mapping; and $\mathscr{U}_k$, a sequence of finite partitions of X by non-empty open and closed subsets such that* $\mathrm{mesh}(\mathscr{U}_k) \to 0$ *as $k \to \infty$. Suppose that there exists a sequence of continuous mappings $\pi_k : Y \to X$ such that $g \circ \pi_k = \pi_k \circ f$ and that $\pi_k(Y) \cap U \neq \emptyset$ for all $U \in \mathscr{U}_k$. Then, there exists a sequence ψ_k $(k = 1, 2, \dots)$ of homeomorphisms from Y onto X such that $\psi_k \circ g \circ \psi_k{}^{-1}$ converges uniformly to f.*

Proof. Let $k \in \mathbf{Z}_+$ be fixed. By assumption, for each $U \in \mathscr{U}_k$, $\pi_k^{-1}(U)$ is a non-empty open and closed subset of Y. Therefore, there exists a homeomorphism $\psi_k : Y \to X$ such that $\psi_k(\pi_k^{-1}(U)) = U$ for each $U \in \mathscr{U}_k$. Because $\pi_k(g(\pi_k^{-1}(U)))) = f(\pi_k(\pi_k^{-1}(U))) \subset f(U)$, $G_{g, \pi_k, \mathscr{U}_k}$ is a subgraph of $G_{f, \mathscr{U}_k}$. Therefore, the conclusion follows from lemma 3.1. □

Let Λ be a two-sided subshift and $x \in \Lambda$. Then, for $k < l$, a word $x_k x_{k+1} \dots x_l$ is said to be j-periodic if $k \le i < i + j \le l$ implies $x_i = x_{i+j}$.

Lemma 3.3 (Krieger's Marker Lemma, (2.2) of M. Boyle [2])**.** *Let (Λ, σ) be a two-sided subshift. Given $k > N > 1$, there exists a closed and open set F such that*

(1) *the sets $\sigma^l(F), 0 \le l < N$, are disjoint, and*
(2) *if $x \in \Lambda$ and $x_{-k} \dots x_k$ is not a j-periodic word for any $j < N$, then*

$$x \in \bigcup_{-N < l < N} \sigma^l(F) \ .$$

Proof. See M. Boyle [2, (2.2)]. □

The next lemma is essentially a part of the proof of the extension lemma given in M. Boyle [2, (2.4)]. The proof essentially follows that of the extension lemma.

Lemma 3.4. *Let (Λ, σ) be a two-sided subshift and (Σ, σ), a mixing two-sided subshift of finite type. Let W be a finite set of words that appear in some elements of Σ. Suppose that Λ is not a finite set of periodic points and that $(\Lambda, \sigma) \xrightarrow{\text{per}} (\Sigma, \sigma)$. Then, there exists a continuous shift-commuting mapping $\pi : \Lambda \to \Sigma$ such that there exists an element $x \in \pi(\Lambda)$ in which all words of W appear as segments of x.*

Proof. Σ is isomorphic to $\Sigma(G)$ for some directed graph $G = (V, E)$. Therefore, without loss of generality, we assume that $\Sigma = \Sigma(G)$. Because $(\Sigma(G), \sigma)$ is a mixing subshift of finite type, there exists an $n > 0$ such that for every pair of elements $v, v' \in V$ and every $m \ge n$, there exists a word of the form $v \dots v'$ of length m. In addition, there exists an element $\bar{x} \in \Sigma(G)$ such that $\bar{x}$ contains all words of W as segments. Let w_0 be a segment of $\bar{x}$ that contains all words of W. Let n_0 be the length of the word w_0. Let $N = 2n + n_0$.

If $v, v' \in V$ and $m \geq N$, then there exists a word of the form $v \ldots w_0 \ldots v'$ of length $m \geq N$. Let $k > 2N$. Using Krieger's marker lemma, there exists a closed and open subset $F \subset \Lambda$ such that the following conditions hold:

(1) the sets $\sigma^l(F), 0 \leq l < N$, are disjoint;
(2) if $x \in \Lambda$ and $x \notin \bigcup_{-N<l<N} \sigma^l(F)$, then $x_{-k} \ldots x_k$ is a j-periodic word for some $j < N$;
(3) the number k is large enough to ensure that if j is less than N and a j-periodic word of length $2k+1$ occurs in some element of Λ, then that word defines a j-periodic orbit which actually occurs in Λ.

The existence of k follows from the compactness of Λ. Let $x \in \Lambda$. If $\sigma^i(x) \in F$, then we *mark* x at position i. There exists a large number $L > 0$ such that whether or not $\sigma^i(x) \in F$ is determined only by the $2L+1$ block $x_{i-L} \ldots x_{i+L}$. If x is marked at position i, then x is unmarked for position l with $i < l < i+N$. Suppose that $x_i \ldots x_{i'}$ is a segment of x such that x is marked at i and i' and that x is unmarked at l for all $i < l < i'$. Then, $i' - i \geq N$. If $x \in \bigcup_{-N<l<N} \sigma^l(F)$, then x is marked at some i where $-N < i < N$. Suppose that $x_{-N+1} \ldots x_{N-1}$ is an unmarked segment. Then, $x \notin \bigcup_{-N<l<N} \sigma^l(F)$, and according to condition (2) $x_{-k} \ldots x_k$ is a j-periodic word for some $j < N$. Suppose that $x_i \ldots x_{i'}$ is an unmarked segment of length at least $2N-1$, i.e. $i' - i \geq 2N-2$. Then, for each l with $i+N-1 \leq l \leq i'-N+1$, $x_{l-k} \ldots x_{l+k}$ is a j-periodic word for some $j < N$. Therefore, it is easy to check that $x_{i+N-1-k} \cdots x_{i'-N+1+k}$ is a j-periodic word for some $j < N$. In this proof, we call a maximal unmarked segment an *interval*. Let $x \in \Lambda$. Let $\ldots x_i$ be a left infinite interval. Then, it is j-periodic for some $j < N$. Similarly, a right infinite interval $x_i \ldots$ is j-periodic for some $j < N$. If x itself is an interval, then it is a periodic point with period $j < N$. If an interval is finite, then it has a length of at least $N-1$. We call intervals of length less than $2N-1$ as *short* intervals. We call intervals of length greater than or equal to $2N-1$ as *long* intervals. If x has a long interval $x_i \ldots x_{i'}$, then $x_{i+N-1-k} \cdots x_{i'-N+1+k}$ is j-periodic for some $j < N$. We have to construct a shift-commuting mapping $\phi : \Lambda \to \Sigma$. Let V' be the set of symbols of Λ. Let $\Phi : V' \to V$ be an arbitrary mapping. Let $x \in \Lambda$. Suppose that x is marked at i. Then, we let $(\phi(x))_i$ be $\Phi(x_i)$. We map periodic points of period $j < N$ to periodic points of Σ. Then, we construct a coding of $\phi(x)$ in three parts. For any $(v, v', l) \in V \times V \times \{N-1, N, N+1, \ldots, 2N-2\}$, we choose a word $\Psi(v, v', l)$ in G of length l such that the word of the form $v\Psi(v, v', l)v'$ is a path in G.

(A) *Coding for short interval:* Let $x_i \ldots x_{i'}$ be a short interval. Then, x is marked at $i-1$ and $i'+1$. We have already defined a code for position $i-1$ and $i'+1$ as $\Phi(x_{i-1})$ and $\Phi(x_{i'+1})$, respectively. The coding for $\{i, i+1, i+2, \ldots, i'\}$ is defined by the path $\Psi(\Phi(x_{i-1}), \Phi(x_{i'+1}), i'-i+1)$.

(B) *Coding for periodic segment:* For an infinite or a long interval, there exists a corresponding periodic point of Λ. The periodic points of Λ are already mapped to periodic points of Σ. Therefore, an infinite or a long periodic segment can be mapped to a naturally corresponding periodic segment.

(C) *Coding for transition part:* To consider a transition segment, let $x_i \ldots x_{i'}$ be a long interval. Then, x_{i-1} has already been mapped to $\Phi(x_{i-1})$

and x_{i+N-1} is mapped according to periodic points. Assume x_{i+N-1} is mapped to v_0. The segment $x_{i-1} \dots x_{i+N-1}$ has length $N+1$. We map the segment $x_i \dots x_{i+N-2}$ to $\Psi(\Phi(x_{i-1}), v_0, N-1)$. In the same manner, the transition coding of right hand side of a long interval is defined. In the same manner, the transition coding of the left or the right infinite interval is defined. It is easy to check that there exists a large number $L' > 0$ such that the coding of $(\phi(x))_i$ is determined only by the block $x_{i-L'} \dots x_{i+L'}$. Therefore, $\phi : \Lambda \to \Sigma$ is continuous. Because Λ is not a set of finite periodic points, there exists an $x \in \Lambda$ such that x contains at least one transition segment or at least one short interval. In the above coding, we can take Ψ such that a short interval or a transition segment is mapped to a word that involves w_0. □

Proposition 3.5. *Let (Σ, σ) be a topologically mixing two-sided subshift of finite type such that Σ is homeomorphic to C. Let (Λ, σ) be a two-sided subshift such that Λ is homeomorphic to C.*

Then, $(\Lambda, \sigma) \rhd (\Sigma, \sigma)$ if and only if $(\Lambda, \sigma) \xrightarrow{\text{per}} (\Sigma, \sigma)$.

Proof. If $(\Lambda, \sigma) \rhd (\Sigma, \sigma)$, then by lemma 2.2, we obtain $(\Lambda, \sigma) \xrightarrow{\text{per}} (\Sigma, \sigma)$. Suppose that $(\Lambda, \sigma) \xrightarrow{\text{per}} (\Sigma, \sigma)$. Without loss of generality, we can assume that $\Sigma = \Sigma(G)$ for some directed graph $G = (V, E)$. We assume that every vertex of V has both at least one outdegree and at least one indegree. Let $k \in \mathbf{Z}_+$. Because (Σ, σ) is topologically mixing, by lemma 3.4, there exists a continuous shift-commuting mapping $\pi_k : \Lambda \to \Sigma$ and $x \in \pi_k(\Lambda)$ such that x contains all words of length $2k+1$ of Σ. Let $\mathscr{U}_k = \{C_{-k}(w) \mid w \in W(2k+1, G)\}$. Then, $\pi_k(\Lambda) \cap U \neq \emptyset$ for all $U \in \mathscr{U}_k$. Because k is arbitrary, by lemma 3.2, we conclude that $(\Lambda, \sigma) \rhd (\Sigma(G), \sigma)$. □

Proof of Theorem 1.1

Proof. If $(\Lambda, \sigma) \rhd (X, f)$, then by lemma 2.2, we obtain $(\Lambda, \sigma) \xrightarrow{\text{per}} (X, f)$. Let $(\Lambda, \sigma) \xrightarrow{\text{per}} (X, f)$ hold. Consider a sequence $\{\mathscr{U}_k\}_{k=1,2,\dots}$ of partitions of X by non-empty open and closed subsets such that $\mathrm{mesh}(\mathscr{U}_k) \to 0$ as $k \to \infty$. Assume $k \in \mathbf{Z}_+$. Let $G_k = G_{f,\mathscr{U}_k}$. Let $\delta > 0$ be such that any $x, x' \in X$ with $d(x, x') < \delta$ are contained in the same element of $\mathscr{U}_k$. Let $\{x_0, x_1\}$ be a δ chain. Let $U, U' \in \mathscr{U}_k$ be such that $x_0 \in U$ and that $x_1 \in U'$. Then, $f(U) \cap U' \neq \emptyset$. Therefore, (U, U') is an edge of G_k. Let $U, V \in \mathscr{U}_k$. Let $x \in U$ and $y \in V$. Because f is chain mixing, there exists an $N > 0$ such that for every $n > N$, there exists a δ chain from x to y of length n. Therefore, for every $n > N$, there exists a path in G_k from U to V of length n. From Lemma 2.3, $(\Sigma(G_k), \sigma)$ is topologically mixing. By lemma 2.7, $\Sigma(G_k)$ is homeomorphic to C. Therefore, there exists a homeomorphism $\psi_k : \Sigma(G_k) \to X$ such that for any vertex u of G_k, $\psi_k(C_0(u)) = u$. By construction, we obtain $G_{\psi_k \circ \sigma \circ {\psi_k}^{-1}, \mathscr{U}_k} = G_{f,\mathscr{U}_k}$. Because $\mathrm{mesh}(\mathscr{U}_k) \to 0$ as $k \to \infty$, by lemma 2.6, we find that $\psi_k \circ \sigma \circ {\psi_k}^{-1} \to f$ as $k \to \infty$. On the other hand, it is easy to verify that $\mathrm{Per}(X, f) \subset \mathrm{Per}(\Sigma(G_k), \sigma)$. By assumption, we obtain $\mathrm{Per}(\Lambda, \sigma) \subset \mathrm{Per}(\Sigma(G_k), \sigma)$. From proposition 3.5, we obtain $(\Lambda, \sigma) \rhd (\Sigma(G_k), \sigma)$. Therefore, by lemma 2.1, we obtain $(\Lambda, \sigma) \rhd (X, f)$. □

Proof of Corollary 1.2

Proof. If a topological dynamical system (X, f) has a fixed point x_0, then $\mathrm{Per}(X, f) = \mathbf{Z}_+$. Therefore, the proof is a direct consequence of theorem 1.1. □

Proof of Corollary 1.3

Proof. Let (Λ, σ) be a two-sided subshift without periodic points. Then, $\mathrm{Per}(\Lambda, \sigma) = \emptyset$. Therefore, from theorem 1.1, the conclusion follows. □

Proof of Corollary 1.4

Proof. A two-sided full shift is chain mixing and has a fixed point. Therefore, the conclusion is a direct consequence of corollary 1.2. □

References

[1] Bowen, R., *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math. **470**, Springer, 1975.

[2] Boyle, M., *Lower entropy factors of sofic systems*, Ergod. Th. & Dynam. Sys. **4** (1984), 541–557.

[3] Kimura, T., *Homeomorphisms of zero-dimensional spaces*, Tsukuba J. Math. **12** (1988), 489–495.

[4] Shimomura, T., *The pseudo-orbit tracing property and expansiveness on the Cantor set*, Proc. Amer. Math. Soc. **106** (1989), 241–244.

[5] Walters, P., *On the pseudo orbit tracing property and its relation to stability*, The Structure of Attraction in Dynamical Systems, Lecture Notes in Math. **668**, Springer-Verlag, Berlin, Heidelberg and New York, 1978, pp.231–244.

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